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# PARAMETER REPEATABILITY FOR ECONOMY. A CASE STUDY OF A SIMPLE LINEAR ELECTRICAL DEVICE

**Antoni Drapella & Tadeusz Niemczyk**

**Summary**

This paper is the first installment of a projected three-part study devoted to both input and output parameter randomness that electrical circuit designers deal with in their work. Parts of the study will differ in the complexity of schemes considered. This paper aims at presenting the methodology while leaving aside the analysis of complex circuits. Therefore, only voltage dividers have been taken into consideration. Four probability distributions of resistor values have been tried: uniform, Gaussian, Laplace and triangular.

**Key words:** Voltage divider, Monte Carlo method.

## Introduction

This paper is devoted to circuits composed of discrete components. As a rule, values of component parameters are random variables differing from their nominal values due to unavoidable production imperfection. Let us call this phenomenon input randomness. This leads to output randomness, i.e. parameters of circuits are random variables too. In other words, input randomness causes output randomness. The term repeatability used in the title and further throughout the paper is complementary to output randomness. Repeatability assessment of electronic devices is a part of their reliability assessment [1]. From a purely mathematical point of view, the investigation of repeatability consists, at least at the beginning, in the derivation of the probability distribution of a function of several random variables. This sometimes covers a multitude of variables. Although probability theory offers tools designed specifically to derive probability densities of functions of several random variables, these are completely inapplicable in practice when we deal with more than two variables. In this situation, the Monte Carlo method proves to be the only solution.

It is common to assume that input randomness is of a Gaussian type, i.e. component parameters follow a normal (Gaussian) distribution. The nominal parameter determines the value of the location parameters. One third of the tolerance interval determines the scale parameter. Assuming a Gaussian (Normal) distribution is nothing else but wishful thinking. For different production factors the actual distribution may depart far from normality. So, in this paper not only the Gaussian but also the Laplace, uniform and triangular distributions have been employed.

## 1. Simple voltage divider

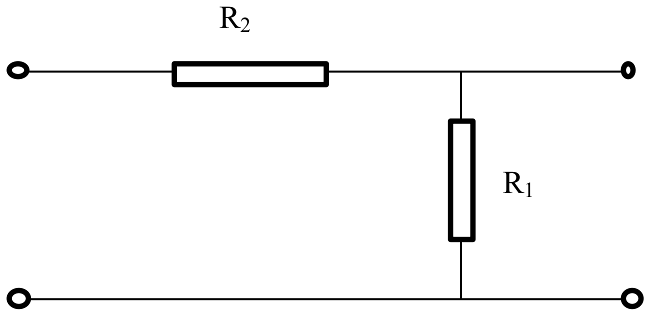


**Figure 1.** A scheme of a simple voltage divider

*Source: own work*

The transfer function has the form:

$$K_u = \frac{R_1}{R_1 + R_2} \quad (1)$$

We assume that, due to the imperfection of production, both $R_1$ and $R_2$ are random variables.

## 2. Probabilistc models employed in this paper

Actual resistance probability distributions are unknown. In this paper we employ four probabilistic models [2] of these unknown distributions. Table 1 explains what particular assumptions really mean.

**Table 1** A list of applied probability distributions.

| Distribution | Meaning | Formula |
|---|---|---|
| Gaussian | Most devices that comprise production batch have their values close to the nominal value. The rest forms long tenuous tails. Greater deviations from nominal value are rare. | (2 -5) |
| Uniform | The opposite of the above distributions. Any deviation from the nominal value, even the greatest, is of the same probability. | (7) |
| Triangle | Although very rarely used in practice it is employed as a reasonable compromise between the Gaussian / Laplace and Uniform distributions. | (8) |
| Laplace | Similar to the Gaussian but more concentrated around the mean values and more tenuous tails | (9) |

The probability density function of the Gaussian distribution. Strictly speaking, it is the doubly censored Gaussian distribution.

$$f(R) = \begin{cases} 0 \; if \; R < R_l \\ \dfrac{1}{C \cdot \sqrt{2 \cdot \pi} \cdot \sigma_R} \cdot \exp\left[ -\dfrac{1}{2} \cdot \left( \dfrac{R - R_o}{\sigma_R} \right)^2 \right] \\ 0 \; if \; R > R_u \end{cases} \quad (2)$$

where $R_l, R_u$ are lower and upper censoring limit, respectively, $C$ is the normalization constant.

$$C = \int_{R_L}^{R_U} \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma_R} \cdot \exp\left[ -\frac{1}{2} \cdot \left( \frac{R - R_o}{\sigma_R} \right)^2 \right] dR \quad (3)$$

$R_o$ is the nominal value

$$R_o = \frac{R_l + R_u}{2} \quad (4)$$

$\sigma_R$ is standard deviation

$$\sigma_R = t_{ol} \cdot \frac{R_o}{3} \quad (5)$$

$t_{ol}$ is resistance tolerance

$$t_{ol} = \frac{R_u - R_l}{2 \cdot R_o} \quad (6)$$

The probability density function of the uniform distribution:

$$f(R) = \begin{cases} 0 \; if \; R < R_L \\ \dfrac{1}{R_u - R_l} \text{ if } R \geq R_l \text{ and } R \leq R_u \\ 0 \; if \; R > R_u \end{cases} \quad (7)$$

The probability density function of the triangular distribution.

$$f(R) = \begin{cases} 0 \; if \; R < R_l \\ \dfrac{2 \cdot (R - R_l)}{(R_u - R_l) \cdot (R_o - R_l)} \text{ if } R \geq R_l \text{ and } R \leq R_o \\ \dfrac{2 \cdot (R_u - R)}{(R_u - R_l) \cdot (R_u - R_o)} \text{ if } R > R_o \text{ and } R \leq R_u \\ 0 \; if \; R > R_u \end{cases} \quad (8)$$

The probability density function of the Laplace distribution:

$$f(R) = \frac{1}{2 \cdot a} \cdot \exp\left( -\left| \frac{R - R_o}{a} \right| \right); \quad a = \frac{R_u - R_l}{6} \quad (9)$$

In this paper, nominal values of resistors $R_o$ are the same and equal to $1000\,\Omega$. As regards a set of tolerances, it contains typical tolerances, namely $t_{ol} = 0.5\%; 1\%; 2\%; 5\%; 10\%; 20\%$.

Moreover, tolerances that appear in practice very rarely, namely $t_{ol} = 3\%$ and $t_{ol} = 15\%$ were added to the set to make figures having $t_{ol}$ as an independent variable smoother. Figure 2 compares visually the distributions employed in this paper. We can also compare distributions in question numerically in terms of the entropy denoted as $E$ which is defined as [1]:

$$E = -\int_{R_l}^{R_u} f(R) \cdot \ln[f(R)] dR \tag{10}$$

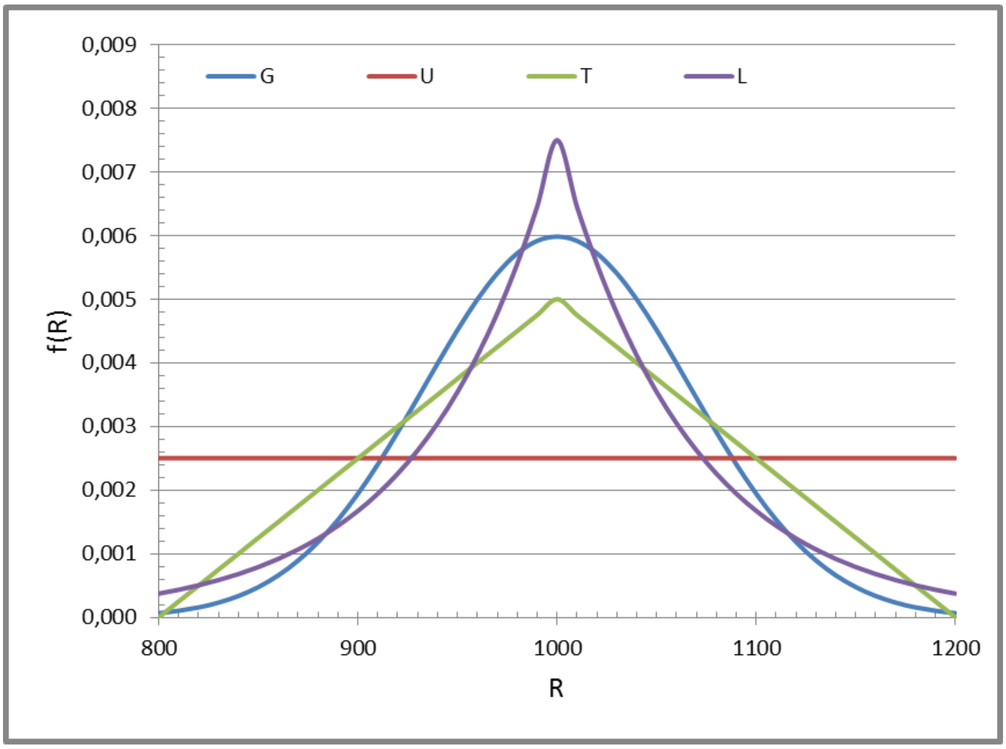


**Figure 2.** Density functions of distributions employed in this paper.

Source: own work.

Table 2 contains entropy formulas that hold for distributions employed.

**Table 2.** Entropy formulas for probability distributions applied.

| Distribution | Formula |
|---|---|
| Normal | $0.5 \cdot \left(1 + \ln_2\left(\frac{2}{9} \cdot \pi \cdot t_{ol}^2 \cdot R_o^2\right)\right)$ |
| Uniform | $\ln_2(2 \cdot t_{ol} \cdot R_o)$ |
| Triangular | $\ln\left(\sqrt{e} \cdot t_{ol} \cdot R_o\right) \cdot \log_2(e)$ |
| Laplace | $\log_2\left(\frac{2}{3} \cdot e \cdot t_{ol} \cdot R_o\right)$ |

Figure 3 shows how the entropies of distributions in question grow when tolerances of resistors are widened.

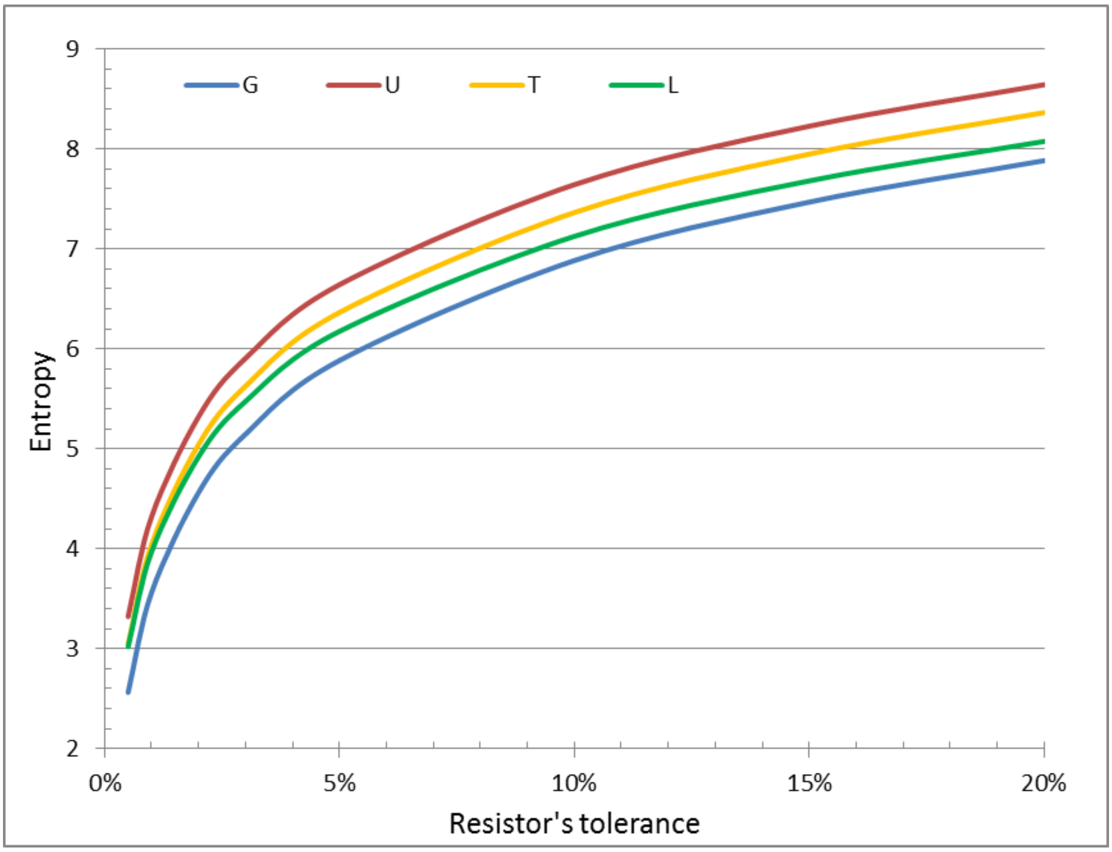


**Figure 3.** The entropy versus tolerances of resistors

Source: own work.

## 3. Generators of random numbers

Let $r_{unf}$ be the random number that follows the uniform distribution located within $<0.1>$ interval.

The normal random values of resistors were generated according the following formula:

$$r_{norm} = R_o \cdot \left[1 + \frac{t_{ol}}{3} \cdot \left( \sum_{v=1}^{12} r_{unf} - 6 \right)\right] \tag{11}$$

The triangular random values of resistors were generated according to the following formula:

$$r_{norm} = R_o \cdot \left[1 + t_{ol} \cdot \left( \sum_{v=1}^{2} r_{unf} - 1 \right)\right] \tag{12}$$

The Laplace random values of resistors were generated according to the following formula:

$$r_{norm} = R_o \cdot \left[1 - \frac{t_{ol}}{3} \cdot \operatorname{sgn}(r_{unf} - 0.5) \cdot \ln(1 - 2 \cdot |r_{unf} - 0.5|)\right] \tag{13}$$

## 4. Modelling the market of devices

Let us imagine that our voltage divider is designed to be mass-produced and sold. Market demand, further denoted as $Dem$, is the probability for a particular device to be sold. Prior to introducing the demand formula, we define some auxiliary variables, namely:

$C_{prod}$ that is the cost of the device to be produced,

$P_{rof}$ that is the projected profit per device,

$C_{ref}$ that is the reference cost which guarantees that demand is equal to one, i.e. all devices will be sold.

The following total cost function is the kernel of the market model:

$$TC = \left( \frac{C_{prod} + P_{rof}}{C_{ref}} \right)^{Mel} - 1 \tag{14}$$

where $Mel$ reflects market flexibility with respect to price. In this simple market, the model price is the sum of the production cost and profit.

The demand formula has been constructed as follows:

$$Dem(TC) = \begin{cases} \exp(-TC) \text{ if TC} > 0 \\ 1 \text{ if } TC \le 0 \end{cases} \tag{15}$$

Figure 4 exemplifies a price-demand relation that follows from (15).

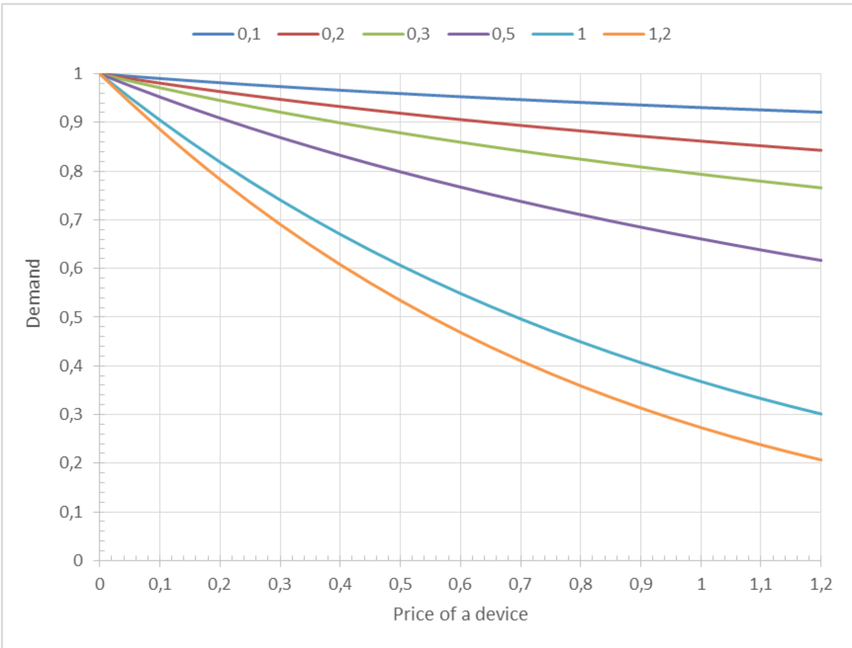


**Figure 4**. Demand for devices versus their price.

Source: own work.

The production cost denoted as $C_{prod}$ has three components

$$C_{prod} = C_{tol}^{R_1} + C_{tol}^{R_2} + C_o \tag{16}$$

where $C_{tol}^{R_1}$ and $C_{tol}^{R_2}$ are the costs of all the efforts made to achieve the required tolerances of $R_1$ and $R_2$, whereas $C_o$ covers all production costs not related to the tolerances.

Tolerance-related costs were defined in the following way:

$$C_{tol}^{R_1} = \left( \frac{t_{ol}^{ref}}{t_{ol}^{R_1}} \right)^{\rho} ; \quad C_{tol}^{R_1} = \left( \frac{t_{ol}^{ref}}{t_{ol}^{R_2}} \right)^{\rho} \tag{17}$$

where $t_{ol}^{ref}$ is the reference tolerance, i.e. such tolerance that costs the least. Further in this paper we assume $t_{ol}^{ref} = 20\%$. In numerical experiments, values of $\rho$ parameter were chosen from $< 0,2 >$ interval.

## 5. Assessing production yield

Let $t_{ol}^{K_u}$ be the required tolerance of the transfer function. Thus,

$$\left(1 - t_{ol}^{K_u}\right) \cdot K_u \le K_u \le \left(1 + t_{ol}^{K_u}\right) \cdot K_u \tag{18}$$

Obviously, the producer wants all the devices to fulfil requirement (18). But some of the devices will not fulfil the requirement due to an unpropitious coincidence of resistor values e.g. both being at the limits of tolerance intervals. The fraction of devices that fulfil (18) is commonly termed yield. Figures 5A – 5D show how resistor tolerances and their distributions impact yield.

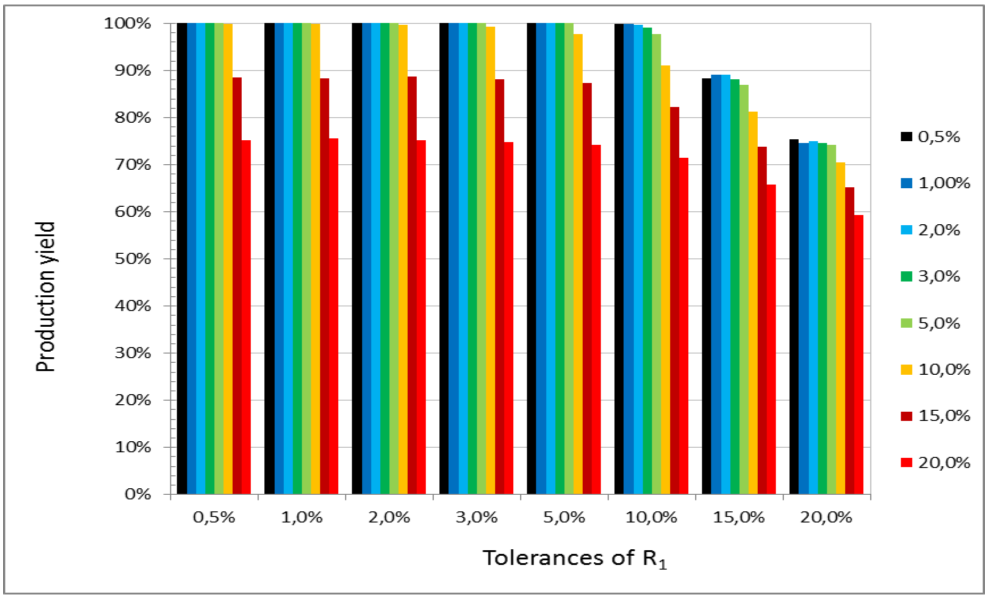


**Figure 5A.** The yield when resistances are normally distributed

Source: own work

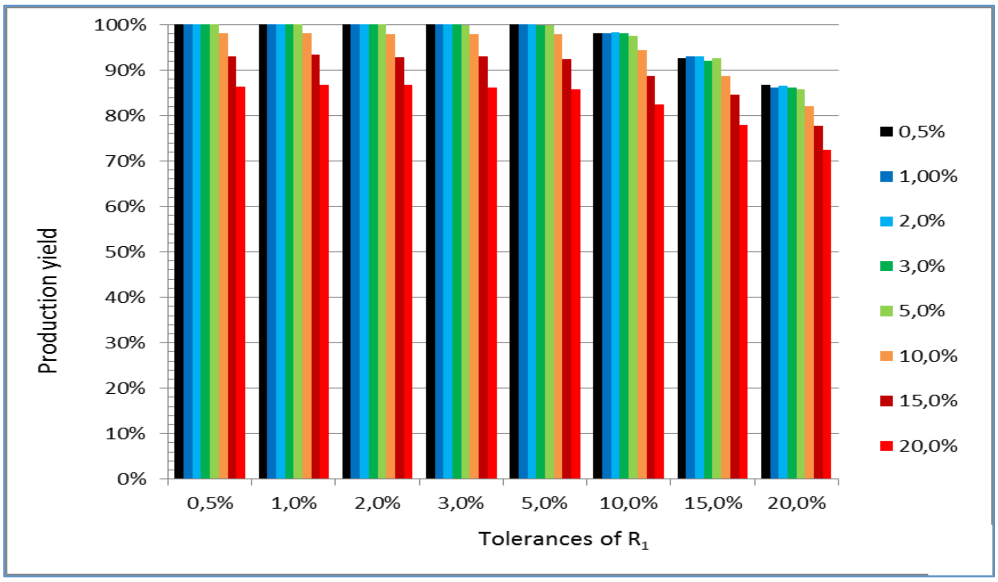


**Figure 5B.** The yield when resistances are distributed according to the Laplace distribution

Source: own work

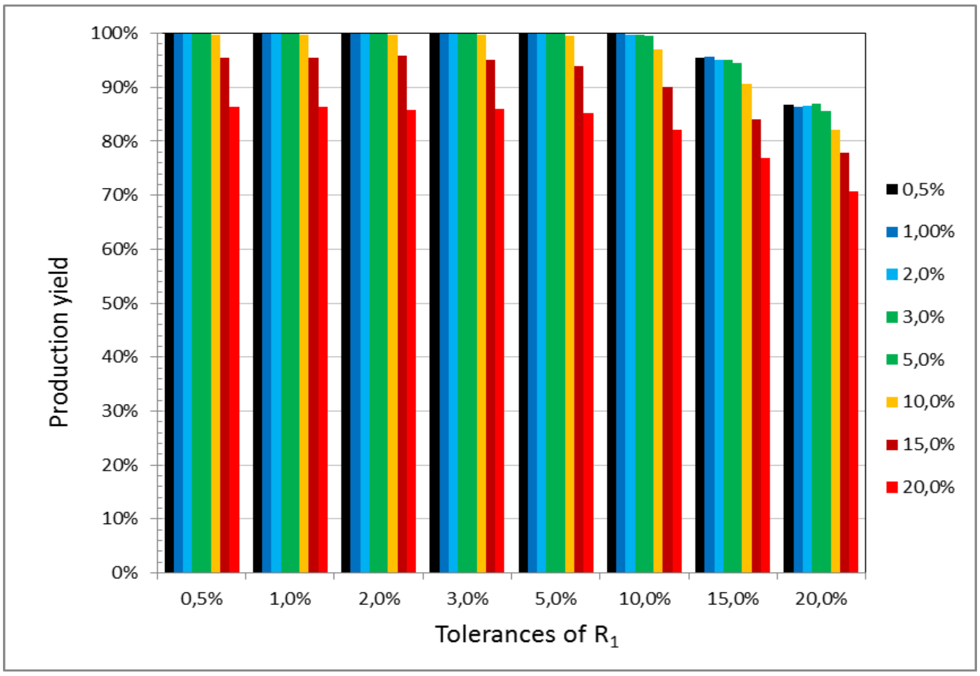


**Figure 5C.** The yield when resistances are triangularly distributed

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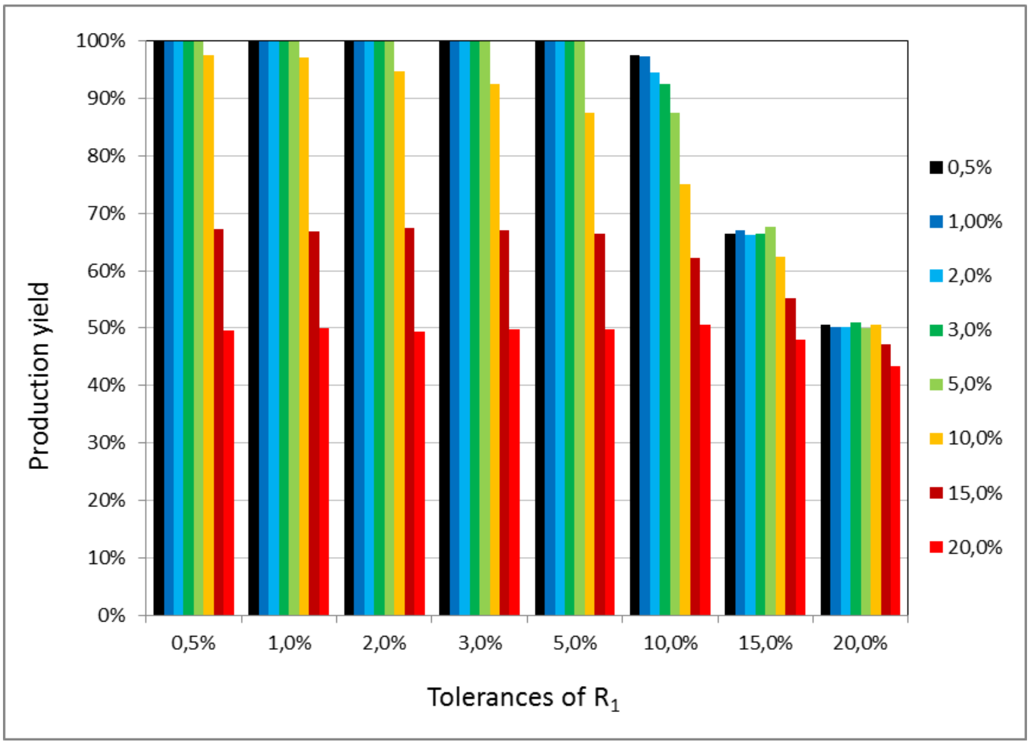


**Figure 5D.** The yield when resistances are uniformly distributed

Source: own work.

## 6. Maximizing producer's income

It is a reasonable step for the producer of devices in question to try to maximize his income, further denoted as $I_{ncome}$. Let us concretize this task and define the income function of the form:

$$I_{ncome} = P_{rof} \cdot Y_{ield} \cdot D_{em}(TC) - C_{prod} \cdot (1 - Y_{ield}) =$$
$$= P_{rof} \cdot Y_{ield} \cdot D_{em}\left(\left(\frac{C_{prod} + P_{rof}}{C_{ref}}\right)^{Mel} - 1\right) - C_{prod} \cdot (1 - Y_{ield}) \tag{19}$$

Combinations of parameter values in (19) form an uncountable set. Table 3 contains input data that were arbitrarily but reasonably chosen for numerical experiments.

**Table 3.** Input data

| Notation | Value | Notation | Value |
|---|---|---|---|
| $C_o$ | 10 | $R_2$ | 1000Ω |
| $C_{ref}$ | 20% | $\rho$ | 1 |
| $Mel$ | 0.5 | $t_{ol}^{Ku}$ | 5% |
| $R_1$ | 1000Ω | $t_{ol}^{ref}$ | 20% |

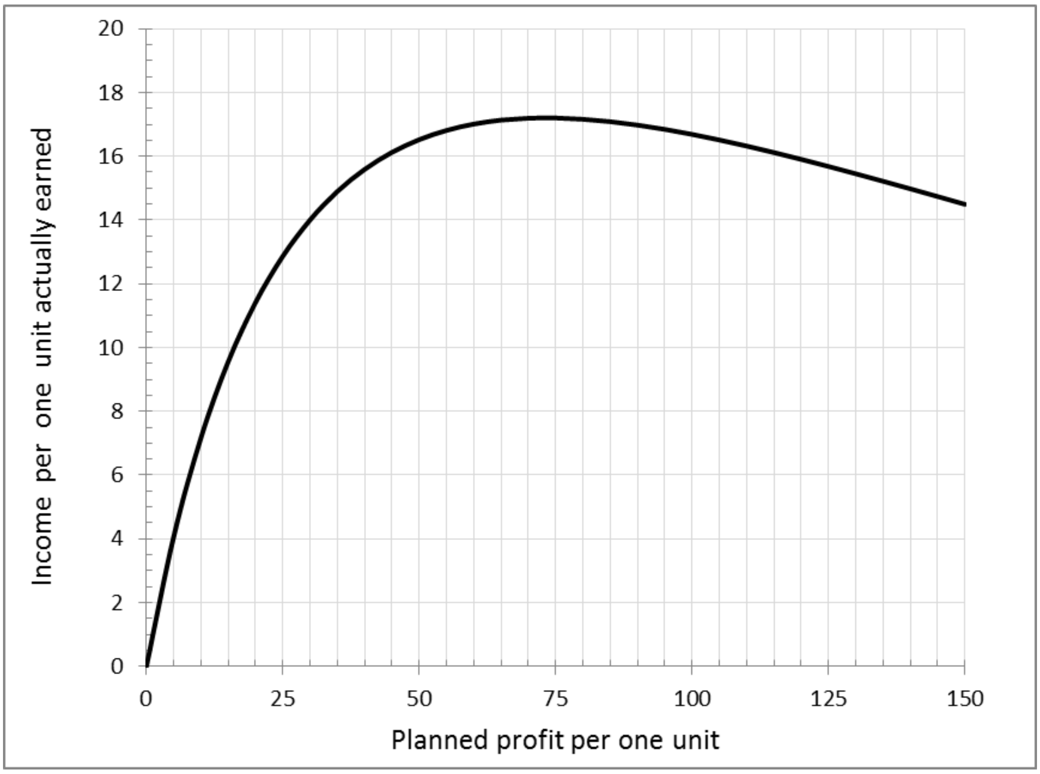


**Figure 6.** Planned profit versus planned income.

Source: own work

## Conclusions

Conclusion 1.

Let us look at set of figures 5 from A to D. They refer to particular distributions and show how tolerances of $R_1$ and $R_2$ impact production yield. As long as tolerances are relatively narrow i.e. $\leq 5\%$, distributions have no noticeable impact on production yield. Widening tolerances causes the yield to drop significantly. So, (for a set of input data collected in Table 3, of course) $5\%$ tolerances for both $R_1, R_2$ components seem to be optimal as balancing the cost of components and the production yield. But this holds only when no market-related factors are taken into account!

Conclusion 2.

All the relations we consider basically change when we embed tolerances and their cost into the market model. Two conditions for devices to be sold are included in the model: a particular device fulfils technical requirements and finds a buyer. Widening tolerances decreases production yield but also decreases the price of the device which, in turn, makes the device more merchantable. In other words, demand for the device increases. Figure 5 shows wanted-versus-earned profit curve and encapsulates what was said above. The curve reaches its maximum at 74 (some imaginary currency units). Relevant optimal tolerances are much wider than in conclusion 1, namely: $t_{ol}^{R_1} = 10\%, t_{ol}^{R_2} = 15\%$.

Conclusion 3.

What was said in Conclusion 2 holds true on the understanding that resistor values follow the normal distribution. Figure 6 shows that a departure from normality has only a slight impact on the producer's income.

## Literature

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2. Encyclopedia of Statistical Science. Kotz S., Read C.B., Balakrishnan N, Vidakovic B. (Chief editors). Willey, 2006.

# POWTARZALNOŚCI PARAMETRÓW WYROBU A JEGO KOSZT WYTWARZANIA NA PRZYKŁADZIE DZIELNIKA NAPIĘCIOWEGO

**Streszczenie**

Niniejszy artykuł jest pierwszym z trzyczęściowego studium poświęconego losowości zarówno wejściowych jak i wyjściowych parametrów wyrobu z jaką projektant styka się w swej pracy. Części wspomnianego studium będą różnić się złożonością rozpatrywanych w nich wyrobów. Niniejszy artykuł ma pokazać metodologię. Analizowanie złożonych układów odłożono na później i rozpatrzono tylko prosty dzielnik napięciowy. Modelowano zakładając cztery rozkłady prawdopodobieństwa rezystorów: Równomierny, Normalny, Laplacea i Trójkątny.

**Słowa kluczowe**: Dzielnik napięciowy, metoda Monte Carlo.

Prof. dr hab. inż. Antoni Drapella
Instytut Uzbrojenia Okrętowego i Informatyki
Wydział Nawigacji i Uzbrojenia Okrętowego AMW w Gdyni
76-200 Słupsk ul. Kotarbińskiego 12/13
adrastat@hotmail.com

Dr inż.Tadeusz Niemczyk
Instytut Uzbrojenia Okrętowego i Informatyki
Wydział Nawigacji i Uzbrojenia Okrętowego AMW w Gdyni
84-230 Rumia ul. Oksywska 8
tniem@wp.pl